Bäcklund Transformations for Fifth-order Painlevé Equations

Ayman Hashem Sakka

Department of Mathematics, Islamic University of Gaza, P. O. Box 108, Rimal, Gaza, Palestine

Reprint requests to Dr. A. H. S.; Fax: (+972)(7)2863552; E-mail: asakka@mail.iugaza.edu

Z. Naturforsch. **60a,** 681 – 686 (2005); received May 25, 2005

In this article we study Bäcklund transformations of the fifth-order Painlevé equations Fif-I, Fif-II, Fif-III and Fif-IV. We derive Bäcklund transformations between these equations and new fifth-order Painlevé equations. The method of derivation is based on the idea of seeking transformations that preserve the Painlevé property. Moreover we give first integrals for the new equations. – MSC2000 classification scheme numbers: 34M55, 35Q51, 37K35

Key word: Bäcklund Transformations.

1. Introduction

One of the important results of the work of the French school on the analytic theory of differential equations at the end of the nineteenth century was the discovery of new special functions, namely the six Painlevé transcendents [1]. They were revealed during the Painlevé classification of the class of second-order differential equations

$$v'' = F(z, v, v'), \tag{1}$$

where F is rational in v and v' and locally analytic in z, that have what today is referred to as the Painlevé property.

The work of Painlevé was extended to higher order equations by several authors, such as Chazy [2], Garnier [3], Bureau [4], Exton [5], Martynov [6] and Cosgrove [7], although no complete classification has yet been given for these higher order equations. It is amongst the equations found in [2 – 7], that equations defining new transcendental functions might be expected to be found. This then leads naturally to the problem of studying the properties of such new equations.

One property that is generally considered to be of particular importance is the existence of Bäcklund transformations (BTs), that is transformations relating a particular Painlevé equation either to itself (with possibly different values of the parameters appearing as coefficients), or to another equation with the Painlevé property. The study of BTs of Painlevé equations has been undertaken by many authors [8 – 17]. One well-known approach is that introduced in [11]. In this approach, an ansatz is made relating the solutions of a Painlevé equation in $v(z)$ to solutions $u(z)$ of a second-order differential equation having the Painlevé property; the ansatz used in [11] relates $v(z)$ and $u(z)$ via

$$(dv^2 + ev + f)u - (v' + av^2 + bv + c) = 0, \tag{2}$$

where a, b, c, e and f are all functions of z only. The algorithm presented in [11] then determines the precise forms of both the BT (2) and the differential equation in $u(z)$, this last by the construction in [11] being at most quadratic in $u''(z)$.

Various generalizations of this approach have since appeared in the literature. In [18, 19] the same ansatz (2) was used to obtain second-order second-degree equations related to $P_I, \ldots, P_{VI}$. In [20], instead of the ansatz (2), the ansatz

$$\begin{aligned} &\left[\left(\Sigma_{i=0}^{2} c_i v^i\right) v' + \Sigma_{i=0}^{4} d_i v^i\right] u \\ &- \left[\left(v'\right)^2 + \left(\Sigma_{i=0}^{2} a_i v^i\right) v' + \Sigma_{i=0}^{4} b_i v^i\right] = 0, \end{aligned} \tag{3}$$

where all a_i, b_i, c_i and d_i are functions of z only, was used to find further second-order second-degree equations related to $P_I, \ldots, P_{VI}$. In [21], (3) was used to obtain second-order fourth-degree equations related to $P_I, \ldots, P_{IV}$.

Meanwhile in [22] it was noted, using as examples P_{III} and P_{IV}, that the ansatz (2) can be used to obtain BTs to equations of a degree higher than two. In [23] a generalized version of the algorithm in [11] was given, allowing the construction of BTs for n-th order equations, in quite a general class, to equations of the same order but perhaps of higher degree; as an example this generalized approach was applied to a particular fourth-order Ordinary Differential Equations (ODE) believed to define a new transcendental function. This generalized algorithm has also been applied in [24] to the fourth-order analogue of P_{I}, and in [25] to the generalized fourth-order analogue of P_{II}. In [26] the approach developed in [22, 23] was applied to P_I and P_{II} to obtain BTs to second-order equations of a degree greater than two. We note that an alternative approach to find BTs appears in [27–29].

Recently, a further generalization of the above approaches is given in [30] in order to obtain BTs for higher order Painlevé equations. The algorithm may be summarized as follows: Assume that we have an n-th order Painlevé-type equation

$$v^{(n)} = f(z,v,v',\ldots,v^{(n-1)}). \tag{4}$$

To study the transformation properties of (4), one may look for transformations of the form

$$\begin{aligned} &H(z,v,v',\ldots,v^{(n-2)})u \\ &-[v^{(n-1)} + G(z,v,v',\ldots,v^{(n-2)})] = 0, \end{aligned} \tag{5}$$

such that $u(z)$ is a solution of another n-th order Painlevé-type equation, in [30] this algorithm to equations of order three, four and six.

In this article, we apply the algorithm to the fifth-order equations

$$v^{(5)} = f(z,v,v',v'',v''',v^{(4)}). \tag{6}$$

Thus we seek a BT of the form

$$H(z,v,v',v'',v''')u - [v^{(4)} + G(z,v,v',v'',v''')] = 0. \tag{7}$$

We assume that G and H in (7) have the following forms:

$$\begin{aligned} G =& (a_{01}v + a_{00})v''' + b_{00}v'v'' + (c_{02}v^2 + c_{01}v + c_{00})v'' + (d_{01}v + d_{00})(v')^2 + (e_{03}v^3 + e_{02}v^2 + e_{01} + e_{00})v' \\ &+ f_{05}v^5 + f_{04}v^4 + f_{03}v^3 + f_{02}v^2 + f_{01}v + f_{00}, \\ H =& (a_{11}v + a_{10})v''' + b_{10}v'v'' + (c_{12}v^2 + c_{11}v + c_{10})v'' + (d_{11}v + d_{10})(v')^2 + (e_{13}v^3 + e_{12}v^2 + e_{11} + e_{10})v' \\ &+ f_{15}v^5 + f_{14}v^4 + f_{13}v^3 + f_{12}v^2 + f_{11}v + f_{10}, \end{aligned} \tag{8}$$

where $a_{ij},\ b_{i0},\ c_{ij},\ d_{ij},\ e_{ij}$ and f_{ij} are functions of z only. In order to simplify the presentation of our results we rewrite the BT (5) as

$$\begin{aligned} v^{(4)} =& (A_1v + A_0)v''' + B_0v'v'' + (C_2v^2 + C_1v + C_0)v'' + (D_1v + D_0)(v')^2 + (E_3v^3 + E_2v^2 + E_1v + E_0)v' \\ &+ F_5v^5 + F_4v^4 + F_3v^3 + F_2v^2 + F_1v + F_0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} &A_j = a_{1j}u - a_{0j},\ j = 0,1, \\ &B_0 = b_{10}u - b_{00}, \\ &C_j = c_{1j}u - c_{0j},\ j = 0,1,2, \\ &D_j = d_{1j}u - d_{0j},\ j = 0,1, \\ &E_j = e_{1j}u - e_{0j},\ j = 0,1,2,3, \\ &F_j = f_{1j}u - f_{0j},\ j = 0,1,2,3,4,5. \end{aligned} \tag{10}$$

Differentiating (9) and using (6) to replace $v^{(5)}$ and (9) to replace $v^{(4)}$, one obtains

$$\begin{aligned} &(\psi_1v' + \psi_2)v''' + \psi_3(v'')^2 + \psi_4v'v'' + \psi_5v'' \\ &+\psi_6(v')^3 + \psi_7(v')^2 + \psi_8v' + \psi_9 = 0, \end{aligned} \tag{11}$$

where ψ_j are polynomials in v with coefficients depending on z,u,u'. In order to find the inverse transformation of (9), one may choose A_j,B_0,C_j,D_j,E_j,F_j so that $\psi_j(j = 0,1,\ldots,8)$ are identically zero. In this case (11) reduces to a polynomial equation in v

$$\psi_9(v) = 0. \tag{12}$$

Solving (12) for v and substituting in (9), one obtains a fifth-order equation for u. The equation for u will be of Painlevé-type if the fourth-order for v defined by the transformation (9) is of Painlevé-type.

2. The Cosgrove's Fif-I Equation

Consider the Cosgrove's Fif-I equation

$$\begin{aligned} v^{(5)} =& 15vv''' + \frac{75}{2}v'v'' - 45v^2v' \\ &+(\lambda z + \alpha)v' + 2\lambda v. \end{aligned} \tag{13}$$

Equation (13) is a group-invariant reduction of the Kaup-Kuperschmidt equation [31]

$$u_{xxxxx} = 15uu_{xxx} + \frac{75}{2}u_x u_{xx} - 45u^2 u_x + u_t, \quad (14)$$

where the invariants are

$$u(x,t) = \begin{cases} (5\lambda)^{2/5} t^{-2/5} v(z), & z = -(5\lambda)^{1/5} t^{-1/5} x - \dfrac{\alpha}{\lambda}, \\ & \lambda \neq 0, \\ v(z), & z = x + \alpha t, \\ & \lambda = 0. \end{cases} \quad (15)$$

When $\lambda = 0$, equation (13) admits the first integral

$$v^{(4)} = 15vv'' + \frac{45}{4}(v')^2 - 15v^3 + \alpha v + \beta, \quad (16)$$

where β is a constant of integration. Equation (16) is labeled in [7] as F-III and solved in terms of hyperelliptic functions.

When $\lambda \neq 0$, one can use a change of variables to write (13) as

$$v^{(5)} = 15vv''' + \frac{75}{2}v'v'' - 45v^2 v' + zv' + 2v. \quad (17)$$

Equation (17) admits the first integral

$$\begin{aligned} &2[v'' - 6v^2 + \frac{2}{3}z][v^{(4)} - 12vv'' - 12(v')^2] \\ &- [v''' - 12vv' + \frac{2}{3}]^2 - 3v[v'' - 6v^2 + \frac{2}{3}z]^2 + K = 0, \end{aligned} \quad (18)$$

where K is a constant of integration. Equation (18) is believed to define a new transcendent.

Now we will apply the algorithm introduced in the introduction to (17). We find that ψ_j in (11) are given by

$$\psi_1 = A_1 + B_0,$$

$$\psi_2 = (A_1 v + A_0)^2 + A_1' v + A_0' + C_2 v^2 + C_1 v + C_0 - 15v,$$

$$\psi_3 = B_0,$$

$$\begin{aligned} \psi_4 = {} & B_0(A_1 v + A_0) + B_0' + 2(D_1 v + D_0) \\ & + 2C_2 v + C_1 - \frac{72}{2}, \end{aligned}$$

$$\begin{aligned} \psi_5 = {} & (A_1 v + A_0)(C_2 v^2 + C_1 v + C_0) + C_2' v^2 \\ & + C_1' v + C_0' + E_3 v^3 + E_2 v^2 + E_1 v + E_0, \end{aligned}$$

$$\psi_6 = D_1,$$

$$\begin{aligned} \psi_7 = {} & (A_1 v + A_0)(D_1 v + D_0) + D_1' v + D_0' \\ & + 3E_3 v^2 + 2E_2 v + E_1, \\ \psi_8 = {} & (A_1 v + A_0)(E_3 v^3 + E_2 v^2 + E_1 v + E_0) \\ & + E_3' v^3 + E_2' v^2 + E_1' v + E_0' + 5F_5 v^4 + 4F_4 v^3 \\ & + 3F_3 v^2 + 2F_2 v^2 + F_1 + 45v^2 - z, \\ \psi_9 = {} & (A_1 v + A_0)(F_5 v^5 + F_4 v^4 + F_3 v^3 + F_2 v^2 \\ & + F_1 v + F_0) + F_5' v^5 + F_4' v^4 + F_3' v^3 + F_2' v^2 \\ & + F_1' v + F_0' - 2v. \end{aligned} \quad (19)$$

Imposing that ψ_j, $(j = 1, \ldots, 8)$ be identically zero, implies that $A_0 = A_1 = B_0 = C_0 = C_2 = D_1 = E_3 = E_2 = E_1 = E_0 = F_5 = F_4 = F_2 = 0$, $C_1 = 15$, $D_0 = \frac{45}{4}$, $F_3 = -15$ and $F_1 = z$. Without loss of generality we may also set $F_0 = u$. The resulting equation $\psi_9 = 0$ then reads

$$v - u' = 0 \quad (20)$$

and the transformation (9) gives

$$u = v^{(4)} - 15vv'' - \frac{45}{4}(v')^2 + 15v^3 - zv. \quad (21)$$

Substituting v from (20) into (21) gives the following fifth-order Painlevé-type equation for u

$$u^{(5)} = 15u'u''' + \frac{45}{4}(u'')^2 - 15(u')^3 + zu' + u. \quad (22)$$

Using the first integral (18) of (17), we find that (22) admits the following fourth-order second-degree equation as a first integral:

$$\begin{aligned} &[u^{(4)} - 12u'u'' + \frac{2}{3}]^2 \\ &= [u''' - 6(u')^2 + \frac{2}{3}z][3u'u''' - \frac{3}{2}(u'')^2 - 12(u')^3 + 2u] \\ &\quad + K, \end{aligned} \quad (23)$$

where K is a constant of integration. Moreover, using the BTs (21) and (20), we find the following BTs between equations (18) and (23):

$$\begin{aligned} v &= u', \\ u &= -\frac{3}{2}vv'' + \frac{3}{4}(v')^2 + 6v^3 \\ &\quad + \frac{(v''' - 12vv' + \frac{2}{3})^2 - K}{2(v'' - 6v^2 + \frac{2}{3}z)}. \end{aligned} \quad (24)$$

3. The Cosgrove's Fif-II Equation

Consider the Cosgrove's Fif-II equation

$$v^{(5)} = 30vv''' + 30v'v'' - 180v^2v' + (\lambda z + \alpha)v' + 2\lambda v. \quad (25)$$

Equation (25) is a group-invariant reduction of the Sawada-Kotera equation [32]

$$u_{xxxxx} = 30uu_{xxx} + 30u_xu_{xx} - 180u^2u_x + u_t, \quad (26)$$

where the invariants are given by (15).

When $\lambda = 0$, (25) admits the first integral

$$v^{(4)} = 30vv'' - 60v^3 + \alpha v + \beta, \quad (27)$$

where β is a constant of integration. Equation (27) is labeled in [7] as F-IV and solved in terms of hyperelliptic functions.

When $\lambda \neq 0$, one can use a change of variables to write (25) as

$$v^{(5)} = 30vv''' + 30v'v'' - 180v^2v' + zv' + 2v. \quad (28)$$

Equation (28) admits the first integral

$$\begin{aligned} &2[v'' - 3v^2 + \frac{1}{12}z][v^{(4)} - 6vv'' - 6(v')^2] \\ &- [v''' - 6vv' + \frac{1}{12}]^2 - 24v[v'' - 3v^2 + \frac{1}{12}z]^2 \\ &+ K = 0, \end{aligned} \quad (29)$$

where K is a constant of integration. Equation (29) is believed to define a new transcendent.

We proceed as in the previous section. For (28) we find that the equation $\psi_9 = 0$ and the transformation (9) read

$$v - u' = 0 \quad (30)$$

and

$$u = v^{(4)} - 30vv'' + 60v^3 - zv, \quad (31)$$

respectively. Substituting v from (30) into (31) gives the following fifth-order Painlevé-type equation for u:

$$u^{(5)} = 30u'u''' - 60(u')^3 + zu' + u. \quad (32)$$

Using the first integral (29) of (28), we find that (32) admits the following fourth-order second-degree equation as a first integral:

$$\begin{aligned} &[u^{(4)} - 6u'u'' + \frac{1}{12}]^2 = 2[u''' - 3(u')^2 + \frac{1}{12}z] \\ &\cdot [12u'u''' - 6(u'')^2 - 24(u')^3 + u] + K_2, \end{aligned} \quad (33)$$

where K_2 is a constant of integration. Moreover, using the BTs (31) and (30) we find the following BTs between (29) and (33)

$$K = K_2,$$

$$v = u', \quad (34)$$

$$u = -12vv'' + 6(v')^2 - 20v^3 + \frac{(v''' - 6vv' + \frac{1}{12})^2 - K}{2(v'' - 3v^2 + \frac{1}{12}z)}.$$

4. The Cosgrove's Fif-III Equation

Consider the Cosgrove's Fif-III equation

$$\begin{aligned} v^{(5)} = {}& 20vv''' + 40v'v'' - 120v^2v' + (\lambda z + \alpha)v' \\ &+ 2\lambda v + \kappa. \end{aligned} \quad (35)$$

The invariants (15) reduce the Lax KdV5 equation [33]

$$u_{xxxxx} = 20uu_{xxx} + 40u_xu_{xx} - 120u^2u_x + u_t + R'(t) \quad (36)$$

with $R'(t) = \begin{cases} -\kappa(5\lambda)^{7/5}t^{-7/5}, & \lambda \neq 0, \\ \kappa, & \lambda = 0 \end{cases}$, to (35).

When $\lambda = 0$, (35) admits the first integral

$$v^{(4)} = 20vv'' + 10(v')^2 - 40v^3 + \alpha v + \kappa z + \beta, \quad (37)$$

where β is a constant of integration. Equation (37) is labeled in [7] as F-V. If $\kappa = 0$, then (37) can be solved in terms of hyperelliptic functions. If $\kappa \neq 0$, then (37) defines a new transcendent.

When $\lambda \neq 0$, one can use a change of variables to write (35) as

$$v^{(5)} = 20vv''' + 40v'v'' - 120v^2v' + zv' + 2v + \kappa. \quad (38)$$

Equation (38) admits the first integral

$$\begin{aligned} &2[v'' - 6v^2 + 4\kappa v + \frac{1}{4}z - 4\kappa^2] \\ &\quad \cdot [v^{(4)} - 12vv'' - 12(v')^2 + 4\kappa v''] \\ &- [v''' - 12vv' + 4\kappa v' + \frac{1}{4}]^2 - 4(2v + \kappa) \\ &\quad \cdot [v'' - 6v^2 + 4\kappa v + \frac{1}{4}z - 4\kappa^2]^2 + K = 0, \end{aligned} \quad (39)$$

where K is a constant of integration. Equation (39) is believed to define a new transcendent.

Applying our method to (38), we find that the equation $\psi_9 = 0$ and the transformation (9) read

$$v - u' = 0 \tag{40}$$

and

$$u = v^{(4)} - 20vv'' - 10(v')^2 + 40v^3 - zv - \kappa z, \tag{41}$$

respectively. Substituting v from (40) into (41) gives the following fifth-order Painlevé-type equation for u:

$$u^{(5)} = 20u'u''' + 10(u'')^2 - 40(u')^3 + zu' + u + \kappa z. \tag{42}$$

Using the first integral (39) of (38), we find that (42) admits the following fourth-order second-degree equation as a first integral:

$$\begin{aligned}&[u^{(4)} - 12u'u'' + 4\kappa u'' + \frac{1}{4}]^2\\ &= 2[u''' - (u')^2 + 4\kappa u' + \frac{1}{4}z - 4\kappa^2]\\ &\quad \cdot [4u'u''' + 2\kappa u''' - 2(u'')^2 - 16(u')^3\\ &\quad - 4\kappa(u')^2 + 8\kappa^2 u' + u + \frac{1}{2}\kappa z + 8\kappa^3] + K_2,\end{aligned} \tag{43}$$

where K_2 is a constant of integration. Moreover, using the BTs (41) and (40), we find the following BTs between (39) and (43):

$$\begin{aligned}&K = K_2, \quad v = u',\\ &u = -2(2v + \kappa)v'' + 2(v')^2 + 16v^3\\ &\quad + 4\kappa v^2 - 8\kappa^2 v - \frac{1}{2}\kappa z - 8\kappa^3\\ &\quad + \frac{[v''' - 12vv' + 4\kappa v' + \frac{1}{4}]^2 - K_2}{2(v'' - 6v^2 + 4\kappa v + \frac{1}{4}z - 4\kappa^2)}.\end{aligned} \tag{44}$$

5. The Cosgrove's Fif-IV Equation

Consider the Cosgrove's Fif-IV equation

$$\begin{aligned}v^{(5)} &= 18vv''' + 36v'v'' - 72v^2v' + 3\lambda v''\\ &\quad + \frac{1}{2}\lambda z(5v''' - 36vv') - \frac{1}{2}\lambda^2 z(2zv' + v)\\ &\quad + \frac{1}{z}[v^{(4)} - 18vv'' - 9(v')^2 + 24v^3 - 3\lambda v' + \kappa].\end{aligned} \tag{45}$$

When $\lambda = 0$, (45) admits the first integral

$$v^{(4)} = 18vv'' + 9(v')^2 - 24v^3 + \alpha v + \kappa z + \beta, \tag{46}$$

where β is a constant of integration. Equation (46) is labeled in [7] as F-VI. Equations F-VI with $\kappa = 0$ can be solved in terms of hyperelliptic functions. When $\kappa \neq 0$, F-VI defines a new transcendent.

When $\lambda \neq 0$, (45) admits the the following fourth-order third-degree equation as a first integral

$$\begin{aligned}&L^3 - \frac{3}{2}\lambda^2 z^2 L[(v + \frac{1}{4}\lambda z)H'' - v'H' + \frac{1}{2}H^2 + \frac{1}{4}\lambda^2 z^2 v]\\ &+ \frac{3}{16}\lambda^3 z^3[(H'')^2 - 4(v + \frac{1}{4}\lambda z)HH'' - 4v(H')^2\\ &+ 4v'HH' - \frac{4}{3}H^3] + \frac{3}{8}\lambda^4 z^3[v'H'' - 4v(v + \frac{1}{4}\lambda z)H'\\ &+ \lambda vH - \frac{1}{4}\lambda(v')^2 + \lambda v(v + \frac{1}{4}\lambda z)^2] + \frac{3}{4}Kz^3 = 0,\end{aligned} \tag{47}$$

where

$$\begin{aligned}&H = v'' - 6v^2 - 2\lambda zv - \frac{1}{8}\lambda^2 z^2,\\ &L = H'' - 6(v + \frac{1}{12}\lambda z)H + 3(v')^2 + \lambda v'\\ &\quad - 12v^3 - 6\lambda zv^2 - \frac{3}{4}\lambda^2 z^2 v + \frac{1}{4}\lambda^2 + \kappa,\end{aligned} \tag{48}$$

and K is a constant of integration. Equation (47) is believed to define a new transcendent.

For (45) the equation $\psi_9 = 0$ and the transformation (9) give

$$v - u' + \frac{1}{4}\lambda z = 0 \tag{49}$$

and

$$\begin{aligned}&\frac{\lambda^2 z}{16}(8u - \lambda z^2)\\ &= v^{(4)} - (18v + \frac{5}{2}\lambda z)v''\\ &\quad - 9(v')^2 - 3\lambda v' + 24v^3 + 9\lambda zv^2 + \lambda^2 z^2 v + \kappa,\end{aligned} \tag{50}$$

respectively. Substituting v from (49) into (50) gives the following fifth-order Painlevé-type equation for u:

$$\begin{aligned}u^{(5)} &= 2(9u' - \lambda z)u''' + 9(u'')^2 - \frac{3}{2}\lambda u''\\ &\quad - 24(u')^3 + 9\lambda z(u')^2 - \lambda^2 z^2 u' + \frac{1}{2}\lambda^2 zu\\ &\quad - \frac{3}{16}\lambda^2 - \kappa.\end{aligned} \tag{51}$$

Using the first integral (47) of (45), we find that (51) admits the following fourth-order second-degree equation as a first integral:

$$(4u' - \lambda z)(H')^2 - H'[(4u'' - \lambda)(H + \lambda u) - 2\lambda u'(4u' - \lambda z)]$$

$$+\frac{4}{3}H^3+2\lambda uH^2-[(6u'-\lambda z)H+G]^2$$
$$-2\lambda^2[(u'-\frac{1}{4}\lambda z)(H+(u')^2)-\frac{1}{4}(u''-\frac{1}{4}\lambda)^2]$$
$$+[(6u'-\lambda z)H+G][4u'(H+\lambda u)-\frac{1}{2}(4u''-\lambda)]$$
$$-\frac{2}{3}\lambda^3u^3+K_2=0, \quad (52)$$

where

$$H=u'''-6(u')^2+\lambda zu',$$
$$G=-\frac{1}{2}u''(6u''-\lambda)+3(u')^2(4u'-\lambda z)+\frac{1}{2}\lambda^2 zu-\frac{3}{16}\lambda^2-\kappa, \quad (53)$$

and K_2 is a constant of integration. Moreover, the BTs (50) and (49) give BTs between (47) and (52).

6. Conclusions

We have applied the algorithm given in [30] to four Painlevé equations of order five, namely Fif-I, Fif-II, Fif-III and Fif-IV equations. For each of these equations we have derived a BT between the given equation and a new Painlevé equation of order five. In addition, we have given a first integral for each of the new equations.

Several generalizations of the used algorithm are possible. For example ,one can start by different forms of G and H in (8). Another generalization may be looking for transformations of the form

$$H(z,v,v',\ldots,v^{(m-1)})u-[v^{(m)}+G(z,v,v',\ldots,v^{(m-1)})], \quad (54)$$

with $m<n-1$.

[1] E. L. Ince, Ordinary Differential Equations, Dover, New York 1956.
[2] J. Chazy, Acta Math. **34**, 317 (1911).
[3] R. Garnier, Ann. Sci. École Normale Sup. **48**, 1 (1912).
[4] F. J. Bureau, Ann. Mat. Pura Appl. (IV) **66**, 1 (1964).
[5] H. Exton, Rend. Mat. **6**, 419 (1973).
[6] I. P. Martynov, Differents. Uravn. **21**, 764 and 937 (1985).
[7] C. M. Cosgrove, Stud. Appl. Math. **104**, 1 (2000).
[8] H. Airault, Stud. Appl. Math. **61**, 31 (1979).
[9] M. Boiti and F. Pempinelli, Nuovo Cimento B **59**, 40 (1980).
[10] A. S. Fokas and Y. C. Yortsos, Lett. Nuovo Cimento **30**, 539 (1981).
[11] A. S. Fokas and M. J. Ablowitz, J. Math. Phys. **23**, 2033 (1982).
[12] Y. Murata, Funkcial Ekvac. **28**, 1 (1985).
[13] K. Okamoto, Ann. Mat. Pura Appl. **146**, 337 (1987).
[14] K. Okamoto, Jpn. J. Math. **13**, 47 (1987).
[15] K. Okamoto, Math. Ann. **275**, 221 (1986).
[16] K. Okamoto, Funkcial Ekvac. **30**, 305 (1987).
[17] V. Gromak, I. Laine, and S. Shimomura, Painlevé Differential Equations in the Complex Plane, de Gruyter, Berlin 2002.
[18] A. Sakka and U. Muğan, J. Phys. A **30**, 5159 (1997).
[19] A. Sakka and U. Muğan, J. Phys. A **31**, 2471 (1998).
[20] U. Muğan and A. Sakka, J. Math. Phys. **40**, 3569 (1999).
[21] A. Sakka, J. Phys. A **34**, 623 (2001).
[22] P. R. Gordoa and A. Pickering, Phys. Lett. A **282**, 152 (2001).
[23] P. R. Gordoa and A. Pickering, J. Math. Phys. **42**, 1697 (2001).
[24] P. R. Gordoa, Phys. Lett. A **287**, 365 (2001).
[25] P. R. Gordoa, Obtaining Bäcklund Transformations for Higher Order Ordinary Differential Equations, preprint 2002.
[26] A. Sakka, Phys. Lett. A **300**, 228 (2002).
[27] P. R. Gordoa, N. Joshi, and A. Pickering, Nonlinearity **12**, 955 (1999).
[28] P. R. Gordoa, N. Joshi, and A. Pickering, Glasgow Math. J. **43A**, 23 (2001).
[29] P. R. Gordoa, N. Joshi, and A. Pickering, Nonlinearity **14**, 567 (2001).
[30] P. R. Gordoa, U. Muğan, A. Pickering, and A. Sakka, Chaos, Solitons and Fractals **22**, 1103 (2004).
[31] D. J. Kaup, Stud. Appl. Math. **62**, 189 (1980).
[32] K. Sawada and T. Kotera, Prog. Theor. Phys. **51**, 1355 (1974).
[33] P. D. Lax, Commun. Pure Appl. Math. **21**, 467 (1968).